Problem set 1

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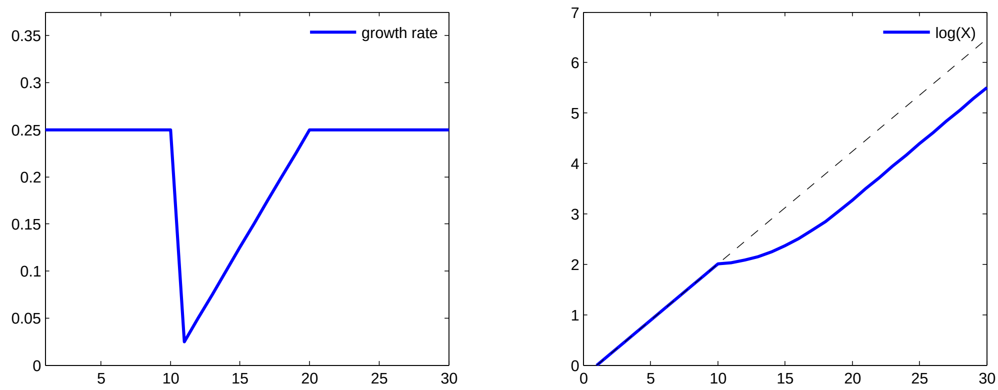
November 21, 2011

Exercise 1 (Romer, 1.2)

Suppose that the growth rate of some variable, X, is constant and equal to $a > 0$ from time $t = 0$ to $t = t_1$; drops to 0 at time t_1; rises gradually from 0 to a from time t_1 to t_2; and is constant and equal to a after t_2.

1. Sketch a graph of the growth rate of X as a function of time.
2. Sketch a graph of $\log X$ as a function of time

Solution



(a) growth rate

(b) $\log(X)$

Exercise 2 (Romer, 1.5)

Consider Solow model with technological and population growth (with growth rates g and n respectively) and depreciation rate δ as we derived it in the class. Suppose $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$.

1. Find expressions for k^*, y^* and c^* as function of the parameters of the model s, n, g, δ and α.
2. What is the golden-rule value of k?
3. What saving rate s^* is needed to yield the golden rule capital stock?

Solution

1. Let's start with $Y = F(K, AL)$ and transform it into the per effective worker form:

$$\begin{aligned} Y &= F(K, AL) \\ y = \frac{Y}{AL} &= \frac{1}{AL}F(K, AL) \\ &= F\left(\frac{K}{AL}, \frac{AL}{AL}\right) = F(k, 1) = k^\alpha 1^{1-\alpha} \\ y &= k^\alpha \end{aligned}$$

Now, using $\dot{K} = -\delta K + sY$ let's find the low of motion for k:

$$\begin{aligned} \dot{k} = \frac{\partial}{\partial t}\frac{K}{AL} &= \frac{\dot{K}AL - K\frac{\partial}{\partial t}(AL)}{(AL)^2} \\ &= \frac{\dot{K}}{AL} - \frac{K}{AL}\frac{\frac{\partial}{\partial t}(AL)}{AL} \\ &= \frac{-\delta K + sY}{AL} - k\frac{\dot{A}L + A\dot{L}}{AL} \\ &= -\delta k + sy - k\left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right) \\ \dot{k} &= sk^\alpha - (\delta + g + n)k \end{aligned}$$

In steady state, the stock of capital per effective worker is stable, hence $\dot{k} = 0$:

$$\begin{aligned} 0 &= sk^\alpha - (\delta + g + n)k \\ sk^\alpha &= (\delta + g + n)k \\ k^{\alpha-1} &= \frac{\delta + g + n}{s} \\ k^* &= \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Note that we have transformed the exponent, so it is positive now (because $\alpha \in (0, 1)$). The steady state output and consumption per effective worker are then

$$\begin{aligned} y^* = (k^*)^\alpha &= \left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} \\ c = \frac{C}{AL} &= \frac{(1-s)Y}{AL} = (1-s)y \\ c^* &= (1-s)\left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

2. The golden rule value of k is such k which maximizes consumption. From the previous we know that we can write $s = (k^*)^{1-\alpha}(\delta + g + n)$, substituting this into the result for consumption yields

$$\begin{aligned} c^* &= (1 - (k^*)^{1-\alpha}(\delta + g + n))\left(\frac{(k^*)^{1-\alpha}(\delta + g + n)}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} \\ &= (k^*)^\alpha - (\delta + g + n)k^* \end{aligned}$$

Intuition can be obtained if we looked consumption as

$$c^* = f(k^*(s)) - (\delta + n + g)k^*(s)$$
$$\frac{\partial c^*}{\partial s} = [f'(k^*(s)) - (\delta + g + n)] \frac{\partial k^*}{\partial s}$$

From this expression we can see that the optimal consumption is obtained if the derivative of the production function at the steady state is equal to $(\delta + g + n)$. *Intuition*: if we want to increase the capital stock by a marginal unit, we will have to pay $(\delta + g + n)$ units to sustain this additional unit of capital.

Furthermore, the condition $f'(k^*(s)) - (\delta+g+n)$ can be rewritten as $\alpha(k^*)^{\alpha-1} = (\delta+g+n)$. From here we can see that

$$k^*_{golden} = \left(\frac{\alpha}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}$$

3. comparing the previous result $k^*_{golden} = \left(\frac{\alpha}{\delta+g+n}\right)^{\frac{1}{1-\alpha}}$ with the optimal capital for a any s, $k^*(s) = \left(\frac{s}{\delta+g+n}\right)^{\frac{1}{1-\alpha}}$, we immediately see that the optimal saving rate is equal to α, $s_{golden} = \alpha$.

Exercise 3

Consider Solow model with population and technological growth $(n,\ g)$. Also, we know that $K_{t+1} = sF(K_t, A_tL_t) + (1 - \delta K)$. In the class we have used the continuous time calculus to derive $\dot{k} = sf(k) - (n + g + \delta)k$.

Working with the discrete variables, show that

$$(1 + g + n)(k_{t+1} - k_t) = sf(k) - (n + g + \delta)k_t.$$

Solution

$$\begin{aligned}
K_{t+1} &= sF(K_t, A_tL_t) + (1 - \delta K) \\
\frac{K_{t+1}}{A_tL_t} &= \frac{sF(K_t, A_tL_t)}{A_tL_t} + \frac{K_t}{A_tL_t} - \delta\frac{K_t}{A_tL_t} \\
\frac{K_{t+1}}{A_{t+1}L_{t+1}} \frac{A_{t+1}L_{t+1}}{A_tL_t} &= sf(k_t) + k_t - \delta k_t \\
k_{t+1}(1 + g)(1 + n) &= sf(k_t) + k_t - \delta k_t
\end{aligned}$$

Now, n and g are growth rate, i.e. the magnitude of these number is about 1e-2., hence the term ng is of magnitude 10e-4 and as such it can be ignored and $(1 + g)(1 + n) \approx (1 + g + n)$

$$\begin{aligned}
k_{t+1}(1 + g + n) &= sf(k_t) + k_t - \delta k_t \\
k_{t+1}(1 + g + n) - k_t &= sf(k_t) - \delta k_t \\
k_{t+1}(1 + g + n) - k_t(1 + g + n) &= sf(k_t) - \delta k_t - k_t(g + n) \\
(k_{t+1} - k_t)(1 + g + n) &= sf(k_t) - (\delta + n + g)k_t
\end{aligned}$$

Note that the right hand side of this equation has the same form as the one we derived for the continuous case of $\dot{k}$.

Exercise 4

In the framework of the dynamic model we have derived in the class, consider a simple modification by adding the government. The demand for output is hence given by private consuption and government expenditures: $Y^D = C + G$.

1. describe the intertemporal government budget constraint
2. analyze the effects of a *permanent* and a *temporary* increase of government spending.

(*hint*: How strong is the wealth effect in either case?)

Solution

1. The traditional way how to model the government is to assume that it has to balance its budgets over infinite horizon (assuming that the interest rate is constant):

$$\sum_{t=0}^{\infty} \frac{G_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{T_t}{(1+r)^t},$$

where G_t denotes government spending in period t is financed by taxes T_t.

Now consider two cases of changes in government spending $\tilde{G}_t = G_t + \Delta G$:

- *permanent*, meaning that $\tilde{G}_t = G_t + \Delta G$ for all t starting today for ever hence this can be written as

$$\begin{aligned} \sum_{t=0}^{\infty} \frac{\tilde{G}_t}{(1+r)^t} &= \sum_{t=0}^{\infty} \frac{T_t}{(1+r)^t} \\ \sum_{t=0}^{\infty} \frac{G_t}{(1+r)^t} + \sum_{t=0}^{\infty} \frac{\Delta G}{(1+r)^t} &= \sum_{t=0}^{\infty} \frac{T_t}{(1+r)^t} \\ \sum_{t=0}^{\infty} \frac{G_t}{(1+r)^t} + \Delta G \frac{1+r}{r} &= \sum_{t=0}^{\infty} \frac{T_t}{(1+r)^t} \end{aligned}$$

  where I used the fact that ΔG is time invariant and hence can be put outside of summation and then used the fact that $\sum_{t=0}^{\infty} q^t = \frac{1}{1-q}$

- *transitory* meaning that $\tilde{G}_t = G_t + \Delta G$ for all $t = 0$ and then $\tilde{G}_t = G_t$ for $t = 1, 2, \ldots$

$$\begin{aligned} \sum_{t=0}^{\infty} \frac{\tilde{G}_t}{(1+r)^t} &= \sum_{t=0}^{\infty} \frac{T_t}{(1+r)^t} \\ \sum_{t=0}^{\infty} \frac{G_t}{(1+r)^t} + \Delta G &= \sum_{t=0}^{\infty} \frac{T_t}{(1+r)^t} \end{aligned}$$

The difference is between the two cases is the difference of $\Delta G \frac{1+r}{r}$ against ΔG. From here you can see that the difference in the discounted value of taxes has to be much stronger in the permanent case. The effect is bigger the smaller the interest rate r is.

2. The budget constraint of the consumer can be written as

$$
\begin{aligned}
\sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} &= \sum_{t=0}^{\infty} \frac{Y_t - T_t}{(1+r)^t} \\
&= \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t} - \sum_{t=0}^{\infty} \frac{T_t}{(1+r)^t} \\
\sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} &= \begin{cases} \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t} - \sum_{t=0}^{\infty} \frac{G_t}{(1+r)^t} - \Delta G \frac{1+r}{r} & \text{for permanent change} \\ \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t} - \sum_{t=0}^{\infty} \frac{G_t}{(1+r)^t} - \Delta G & \text{for transitory change} \end{cases}
\end{aligned}
$$

and hence the wealth effect of the permanent change in the consumer wealth is much stronger for the *permanent* change in the government spending for the same reasons as argued above.