Diploma Macro Classes

Solow model and dynamic ISLM

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Housekeeping

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Plan for today

1. Solow model
   - overview
   - model derivations
   - experiments with parameter shifts
2. dynamic ISLM model
3. Exercises for the next week

Part I

Solow Model

1. Overview
   - Continuous and Discrete time
   - Overview
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   - Exogenous variables
   - aggregate variables
3. Parameter experimets
   - Experiments with model parameters and exogenous variables
   - Other possible experiments

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Growth Models

Any economic model has to describe

- how are final goods produced - production function
- how are final good utilised (consumtion, saving)
- what is the dynamics of the factors of production

In macroeconomics, observed data is discrete:

- CPI is published monthly
- GDP quarterly
- budget deficits anually

but step function are not continuous (what about f'?), so continuous time more convenient.

Timing

Continuous vs. discrete time

We are interested in changes variables over time. Consider variable $X(t)$ for continuous time, what is the time derivative?

$$\dot{X} \equiv \frac{d}{dt}X(t) = \lim_{\tau \to 0} \frac{X(t+\tau) - X(t)}{\tau}$$

This is the same as $X_{t+1} - X_t$ if the lenght of the time period goes to zero.
We are also interested in growth rates: $\frac{X_{t+1} - X_t}{X_t}$
Convenient trick to get growth rates, use logarithm and then time differentiate:

$$\frac{d}{dt}\log(X(t)) = 1/X \frac{d}{dt}X(t) = \dot{X}/X$$

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In Solow model

- Production function: $Y(t) = F(K, AL)$
- Output demand: $Y(t) = C(t) + S(t)$
- the savings function $S(t) = sY(t)$
- Capital dynamics $\dot{K}(t) = -\delta K(t) + S(t)$
- exogenously given
  - technology $A(t) = A_0 \exp(gt)$
  - population $L(t) = L_0 \exp(nt)$

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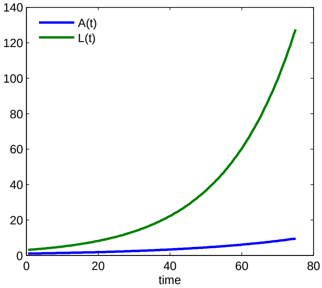
Exogenous variables: variables which are determined outside of the model, here

- technology $A(t) = A_0 \exp(gt)$
- population $L(t) = L_0 \exp(nt)$

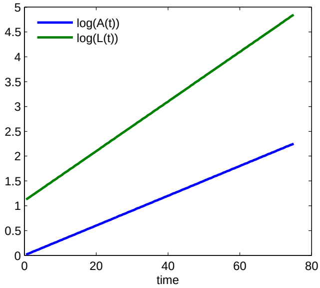
Notation: $\frac{d}{dt}X(t) = \dot{X}(t) = \dot{X}$
Growth rate of variable X: $\dot{X}/X$
So what is the growth rate of population?

$$\dot{A}/A = \frac{d}{dt}\log(A(t)) = \frac{d}{dt}[\log(A_0 \exp(gt))]$$
$$= \frac{d}{dt}[\log(A_0) + \log(\exp(gt))] = \frac{d}{dt}[\log(A_0)] + \frac{d}{dt}[gt]$$
$$\dot{A}/A = g$$

And similarly $\dot{L}/L = n$



(a) original variables: exponential functions



(b) logs: linear functions

Figure: Setting $A_0 = 1$, $L_0 = 3$, $g = 0.03$, $n = 0.02$

Interpretation: if we take logarithms, then the slope is equal to the growth rate

Assumption: production function $F(\cdot, \cdot)$ is homogeneous of degree one

Definition

function g is homogenous of degree k if

$$g(\gamma \boldsymbol{v}) = \gamma^k g(\boldsymbol{v}),$$

$\boldsymbol{v}$ is possibly a vector.

In economics: Cobb-Douglas: $Y = F(K, AL) = K^\alpha (AL)^{1-\alpha}$
checking the condition:

$$\begin{aligned} F(\gamma K, \gamma AL) &= (\gamma K)^\alpha (\gamma AL)^{1-\alpha} = \gamma^\alpha K^\alpha \gamma^{1-\alpha} (AL)^{1-\alpha} = \gamma K^\alpha (AL)^{1-\alpha} \\ &= \gamma F(K, AL) \end{aligned}$$

Production function: $Y = K^{\alpha}(AL)^{1-\alpha}$
Interpretation:

- economy big enough so economies of scale are exhausted, no further specialisation possible
- anything else (land, human capital,...) relatively unimportant

Realistic? (depending on the research question), but extremely useful: it allows up to solve the model in the per effective worker terms ($y = Y/(AL), k = K/(AL)$):

$$Y = F(K, AL)$$
$$y \equiv \frac{1}{AL} Y = \frac{1}{AL} F(K, AL) = F\left(\frac{K}{AL}, \frac{AL}{AL}\right) = F(k, 1) \equiv f(k)$$
$$y = f(k)$$

Interpretation: dividing economy into AL segments, each segment producing $\frac{1}{AL}$ fraction of total output.

Potential variable of interest:

- output Y
- output per capita (per worker) Y/L
- output per effective worker Y/AL

We already defined $Y/AL = y = f(k)$ - capital per effective worker is the state variable, i.e. the only variable needed to know in order to solve for (optimal) choices of other variables (output, consumption, savings)

Way how to solve the model:

1. solve the model for k
2. get the other variables:
   - per effective worker $y = f(k)$, $c = (1-s)f(k)$
   - per capita $Y/L = f(k)A$, $C/L = (1-s)f(k)A$
   - aggregate $Y = f(k)AL$, $C = (1-s)f(k)AL$

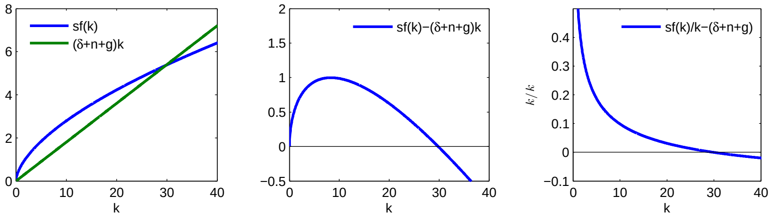
Solving the model

Law of motion for effective capital

We know $\dot{K}(t) = -\delta K(t) + S(t)$, but we are interested in $\dot{k}$:

$$
\begin{aligned}
\dot{k} &= \frac{d}{dt}\frac{K}{AL} = \frac{\frac{d}{dt}[K]AL - K\frac{d}{dt}[AL]}{(AL)^2} \\
&= \frac{\dot{K}AL - K(\dot{A}L + A\dot{L})}{(AL)^2} = \frac{\dot{K}}{AL} - \frac{K}{AL}\left(\frac{\dot{A}L}{AL} + \frac{A\dot{L}}{AL}\right) \\
&= \frac{-\delta K + S}{AL} - k\left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right) \\
&= -\delta k(t) + \frac{sF(K, AL)}{AL} - k\,(g + n) \\
\dot{k} &= sf(k) - (\delta + g + n)k
\end{aligned}
$$

Steady state: $\dot{k} = 0$ so $sf(k) = (\delta + g + n)k$



(a) focusing on $\dot{k}$ (b) phase diagram (c) focusing on $\dot{k}/k$

Figure: Setting $\alpha = 0.6$, $f(k) = k^{\alpha}$, $\delta = 0.1$, $s = 0.7$,

Interpretation: There are two steady states $k*$:

- trivial $k^* = 0$ nothing is produced, no depreciation
- $k^* = 47.0454$

$\dot{k} \approx k_{t+1} - k_t$, hence $k_{t+1} = sf(k) - (\delta + g + n - 1)k$

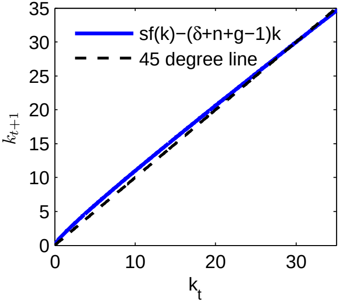
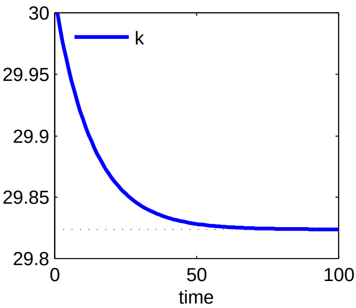
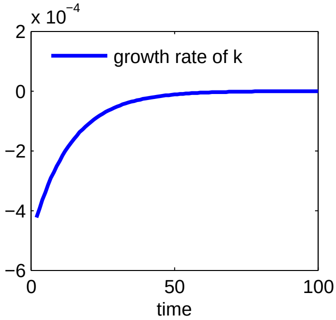


Figure: Discrete time case

Interpretation: if $k_{t+1} > k_t$, then capital stock is growing, if $k_{t+1} < k_t$ capital stock is shrinking. The steady state is when $k^* = k_{t+1} = k_t$, hence the intersection with the 45 degree line.



(a) capital per effective worker k



(b) growth rate of k

Figure: Assume starting values $A_0 = 1$, $L_0 = 3$ and $K_0 = 5$

Interpretation: k converges towards k^*, but the growth rate is slowing down and eventually get to 0

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Rescalling

Now we have described the behavior of k, but we are ultimately interested in the behavior of K, Y, C, or C/L.
Recall, $A_t = A_0 \exp(gt)$ and $L_t = L_0 \exp(nt)$ exogenously given by starting values A_0 and L_0. Assume that we also have K_0. How will the economy evolve?
Plan:

- describe behavior of k
- using k, get K
- using K, A and L, get Y and C and Y/L

Trick: we are interested in growth rate $p = \dot{P}/P$ of P and we know

- $P = Q^{\beta} R^{\gamma}$
- growth rate of Q is q, meaning $q = \dot{Q}/Q = \frac{d}{dt}\log(Q(t))$
- growth rate of R is r, meaning $r = \dot{R}/R = \frac{d}{dt}\log(R(t))$

then use the trick with logarithms again:

$$p = \dot{P}/P = \frac{d}{dt}\log(P(t)) = \frac{d}{dt}\log(Q^{\beta}R^{\gamma}) = \frac{d}{dt}[\beta\log(Q) + \gamma\log(R)]$$

$$= \beta\frac{d}{dt}\log(Q) + \gamma\frac{d}{dt}\log(R)$$

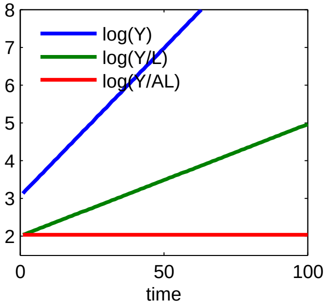
$$p = \beta q + \gamma r$$

What are the steady state growth rates?

- capital per effective worker: $k(t) = k^*$, hence $\dot{k}/k = 0$
- output per effective worker: $y = k^*$, so $y(t) = y^* = (k^*)^\alpha$, hence $\dot{y}/y = \alpha\dot{k}/k = \alpha 0 = 0$
- output per capita: $\tilde{Y} \equiv Y/L = A\frac{Y}{AL} = Ay$, hence $\dot{\tilde{Y}}/\tilde{Y} = \dot{\tilde{A}}/\tilde{A} + \dot{y}/y = g + 0 = g$
- aggregate output: $Y = L\tilde{Y}$, hence $\dot{Y}/Y = \dot{L}/L + \dot{\tilde{Y}}/\tilde{Y} = n + g$

Time paths

Growth rate of different outputs



Interpretation: in the steady state (effectively after first 60 periods, remember we started with $k_0 < k^*$),

- the output per effective worker does not grow
- output per capita grows at the grow rate of technology g
- aggregate output grows at rate $g + n$

(slopes)

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Effects of changes in parameters and exogenous variables

Scenario: Change in population growth

After fall of communism in 1989, young people got new opportunities to travel, work and study abroad and focus on their careers $\rightarrow$ young couples are postponing having children.
How can you model this in the framework we developed?

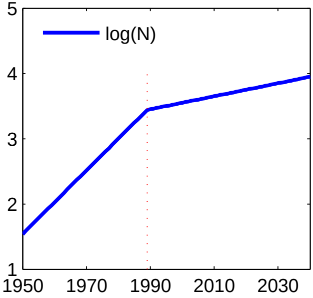
- n decreases to n_{new}, so $A(t) = A(1989)\exp(n_{new}t)$ for $t > 1989$
- what is the effect on steady state capital per effective worker?

$$\dot{k}_{new} = sf(k) - (\delta + g + n_{new})k$$

Will the new steady state capital k^*_{new} be lower or higher then the old steady state k^*?

Changes in parameters

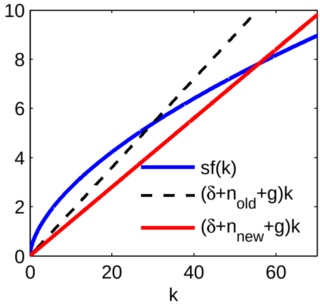
Scenario: Change in n



Interpretation: population growth rate decreased in 1989

Changes in parameters

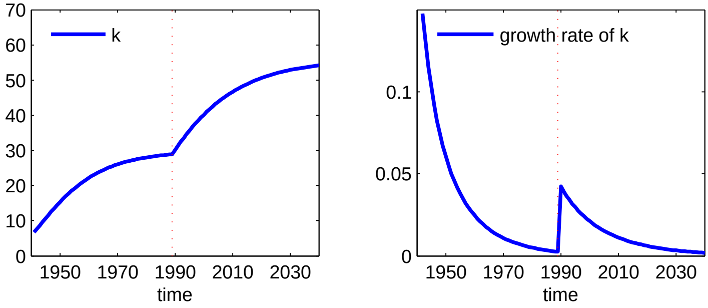
Scenario: Change in n



Result: $k^*_{new} > k^*_{old}$, but still $\dot{k} = 0$ in the new steady state
Interpretation: with lower n, less capital is needed every period to equip new people (recall dividing the economy into AL units)

Changes in parameters

Scenario: Change in n

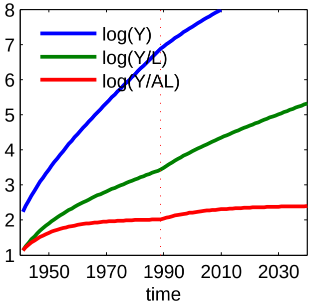


Transition

- assuming starting from the old steady state:
  starting $\dot{k}$ is zero and eventually $\dot{k}$ is zero again
  we also know that $k^*_{new} > k^*_{old}$, hence in the transition $\dot{k} > 0$
- here little more complicated as we start in 1940 with $k > k^*_{old}$,
  hence the two transition periods

Changes in parameters

Scenario: Change in n



Answer: What output? - focus on the steady state

- output per effective worker y grows at the same rate as k, (=0), no change
- output per worker $(= yA)$ grows at the same rate g, no change
- aggregate output $(= yAL)$ grows at the same rate $g + n_{new}$, decrease

What about the transition period?

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Possible scenarios

What might change in the model:

- parameters
  - δ, n, g, s (...,open borders mean higher immigration $\rightarrow g \uparrow$,...)
- Exogenous variables
  - L_t, A_t: (famine and wars can decimate the population)
- capital stock K, (natural disaster, flooding)

Possible question: describe the new steady state and the transition period after a flooding/war/great new discovery...

Part II

dynamic ISLM

Temporary vs permanent shocks
Production function
Labour market
Goods market
Permanent vs temporary shock

Common problems with dynamic macro models

Common problem: understand the different implications of temporary and permanent shocks
Aim of this part: Go over the implication of the shock in productivity in detail and compare the effects of permanent and temporary shock.

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different effects for the permanent wealth of the consumers

- permanent effect: strong wealth effect
- temporary effect: weak wealth effect

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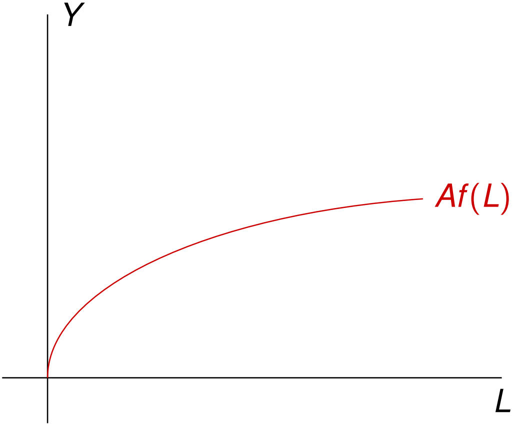
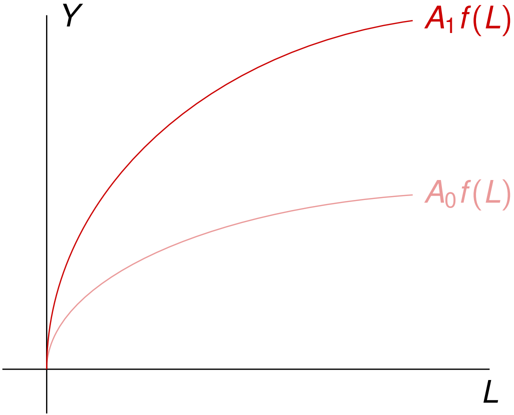


Production function, ignore capital for the moment, $Af(L)$ could be for example $A(L^{0.6})$ - (concave function)



positive technological shock: $A \uparrow \rightarrow$ production function rotating upwards

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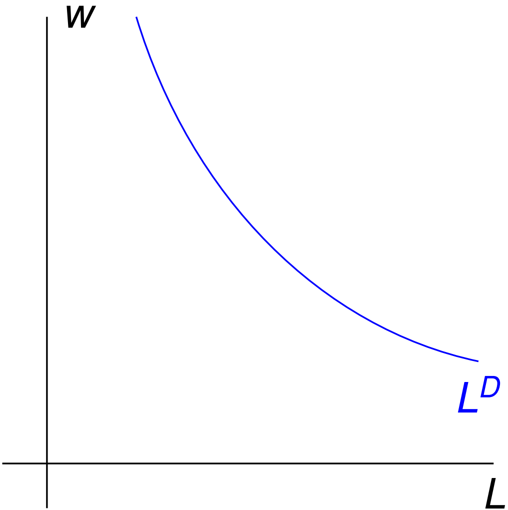
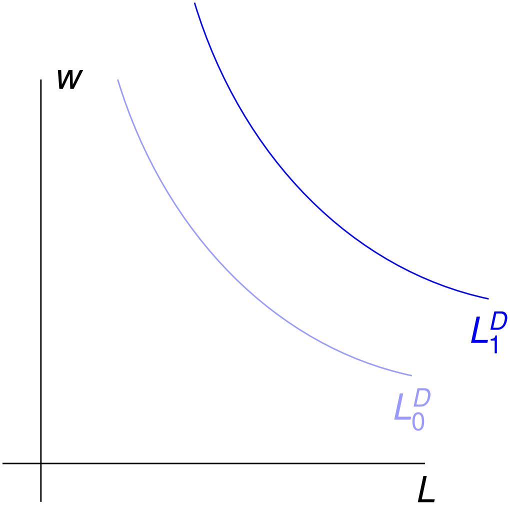
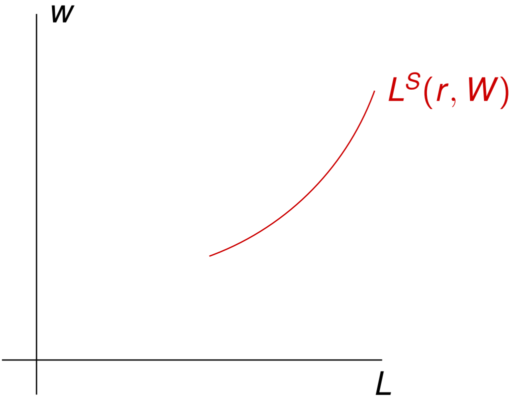


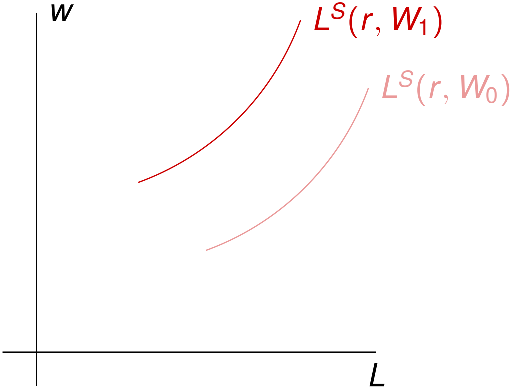
Labour demand: firms hiring worker as long as $MP_L w \geq w$



Productivity shock: $MP_l = \partial Y / \partial L = \partial Af(L) / \partial L$, hence if $A \uparrow$, then $L^D \uparrow$



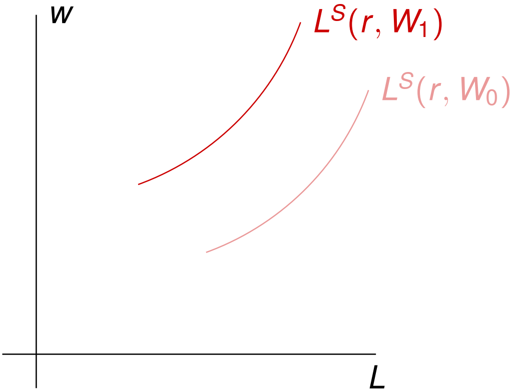
Consumers are balancing MC and MB of leisure → disutility of work is increasing function, so to work more I have to be compensated more → L^S upwardsloping



Wealth effect: more wealth→consume more of both consumption goods and leisure → reduce labour supply (for the same wage willing to only work less)

Labour market

Labour supply - interest rate changes



intertemporal leisure substitution $r_1 < r_0$ (keeping w and w' constant):
lower interest rates→ borrow leisure, work less today and more tomorrow
(Williamson, 3rd edition, page 316)

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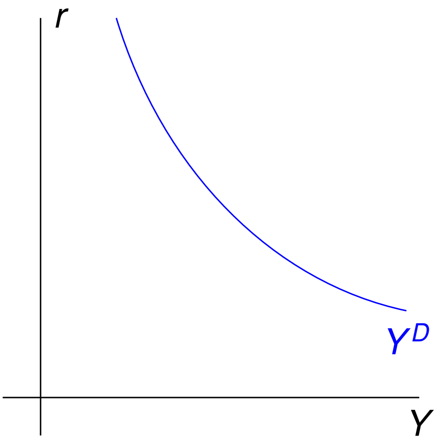
Goods market

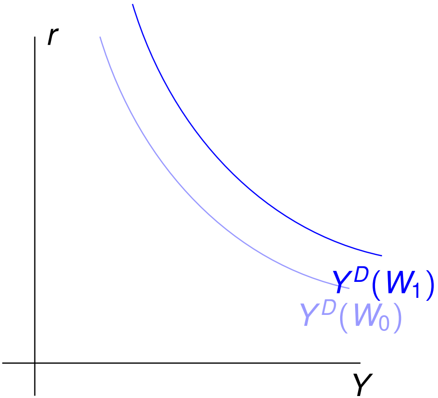
Demand for output



Intertemporal consumption substitution: lower interest rate induces consumers to consume more today at expense of tomorrow

Goods market

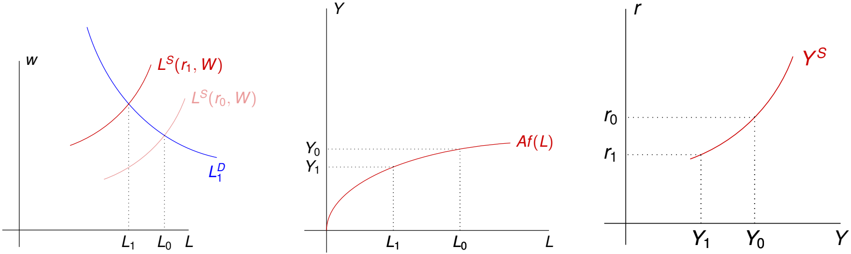
Demand for output - wealth changes



Wealth effect: higher wealth induces higher consumption today, as well as in the future ($W_0 < W_1$)

Goods market

Demand for output - wealth changes



Labour market — Production function — goods market

suppose $r_1 < r_0$, then labour supply shift leftwards→ labour market clears at lower labour → movement along the production function → less output produced → hence r_1 corresponds to lower output →the output supply curve has positive slope

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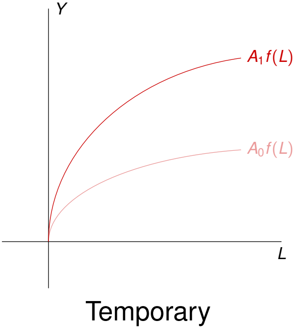
Permanent vs temporary tech. shock

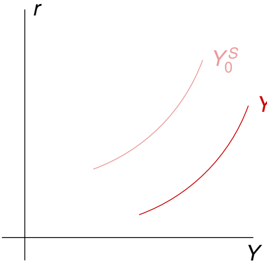
Production function



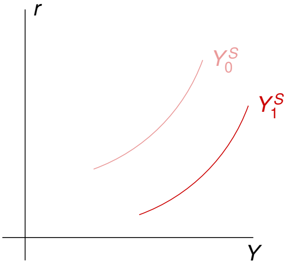
The effect of the change in A is the same for $Y = Af(L)$ (keeping L constant)

Permanent vs temporary tech. shock

Supply of output, effect of A



Temporary

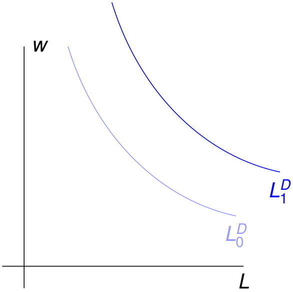


Permanent

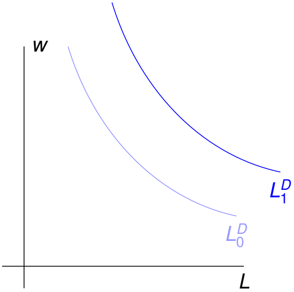
The effect of the change in A is the same for Y^S (keeping r constant)

Permanent vs temporary tech. shock

Demand for labour



Temporary

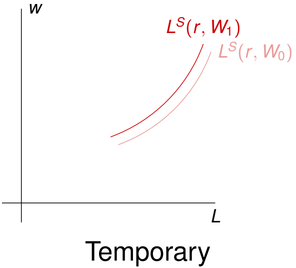


Permanent

The effect on MP_L is the same in the current period (keeping w constant)

Permanent vs temporary tech. shock

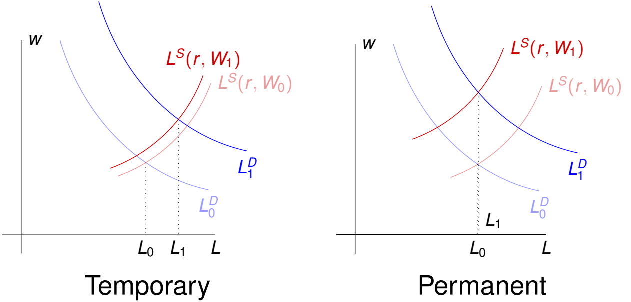
Supply for labour



Wealth effect is negligible if the shock is only temporary

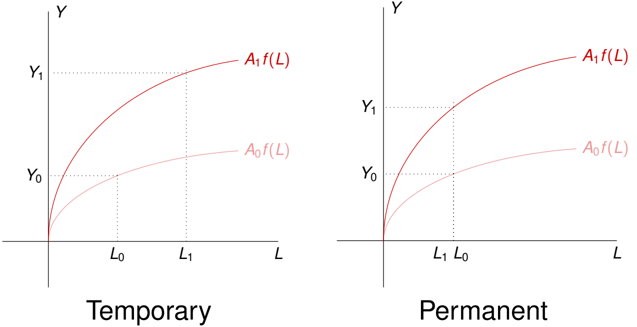
Permanent vs temporary tech. shock

Labour market clearing



Permanent vs temporary tech. shock

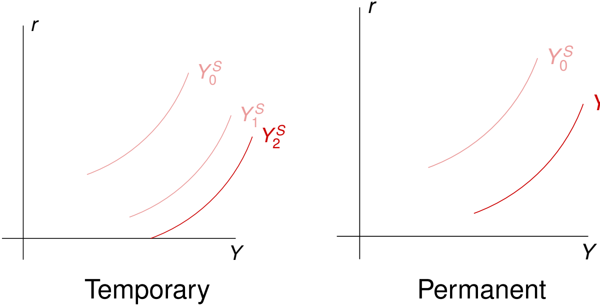
Labour market spillovers into production function



Two effect: shift of the production function AND shift along the production function

Permanent vs temporary tech. shock

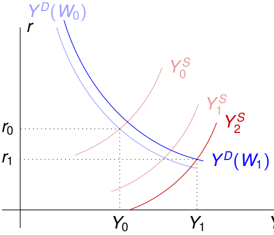
Spillovers from production function to Y^S



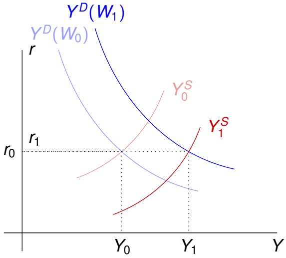
Higher labour supply shifts Y_2^S even further in the temporary A scenario

Permanent vs temporary tech. shock

Goods market clearing



Temporary



Permanent

Higher labour supply shifts Y_2^S even further in the temporary A scenario.
Secondary effect: lower interest rate in the temporary case shifts L^S somewhat back, so Y^S shifts back a little (very small, not shown)